Applications of Graphic Statics to the Plastic Design of Reinforced Concrete Structures

Conference Paper · September 2022

3 authors, including:

Marina Konstantatou
Foster + Partners

Pierluigi D'Acunto
Technische Universität München

Proceedings of the IASS 2022 Symposium affiliated with APCS 2022 conference
Innovation·Sustainability·Legacy
19 – 22 September 2022, Beijing, China

Applications of Graphic Statics to the Plastic Design of Reinforced Concrete Structures

Marina KONSTANTATOU*, Miguel FERNÁNDEZ RUIZ[a], Pierluigi D'ACUNTO[b]

* Specialist Modelling Group,
Foster + Partners, London, UK
mk822@cam.ac.uk
[a] Universidad Politécnica de Madrid, Spain
[b] Professorship of Structural Design, Technical University of Munich, Germany

Abstract
Graphic statics can be used as a design and analysis tool to develop discrete rigid-plastic stress fields and their underlying strut-and-tie models in reinforced concrete structures. This approach relies on the construction of reciprocal Airy Stress Functions (ASF) to generate form and force diagrams; these diagrams are subsequently combined into Minkowski Sums (MS), from which discrete rigid-plastic stress fields are derived. This paper focuses on the design of discrete rigid-plastic stress fields for 2D practical case studies, including typical reinforced concrete structural elements with various boundary conditions and loadings. To this end, a two-fold framework is introduced that comprises: a) the generation of an initial strut-and-tie model based on a stress field produced via an elastic-plastic Finite Element Analysis (FEA) for given boundary and loading conditions on a reinforced concrete structural element; b) the transformation of such strut-and-tie model using non-linear optimisation methods to generate a geometrically and statically admissible discrete rigid-plastic stress fields. This approach provides structural engineers with novel, visual, and intuitive methods for designing and analysing reinforced concrete structures.

Keywords: Airy stress function, graphic statics, reciprocal diagrams, reinforced concrete, stress fields, structural analysis and design, strut-and-tie models, plasticity theory

1. Introduction
By taking advantage of the reciprocity between form and force diagrams [1][2][3][4], graphic statics enables the analysis and design of structures in static equilibrium. Over the last few years, several methods and tools based on graphic statics have been developed that allow dealing with three-dimensional structures [5][6][7][8][9]. In fact, graphic statics' form and force diagrams can be regarded as projections of higher-dimensional Airy Stress Functions [2][10][11][12]. Moreover, form and force diagrams can be combined in one single geometric construction, the Minkowski Sum [13]. The latter can then be used to generate discrete stress fields [14].

This research aims to introduce a holistic and interactive graphic statics framework for designing structural concrete elements based on the parametrisation of the ASF. This framework can be helpful not only as a theoretical approach but also as a tool for practitioners. To this end, the theoretical approach and methods for design and analysis of discrete rigid-plastic stress fields - and their associated strut-and-tie models presented in previous research work [12][14][15] - are applied here to typical practical case studies of reinforced concrete structural elements; namely, a wall with an opening and a beam with variable cross-section.

2. Methods

The proposed method for the design of discrete stress fields in reinforced concrete structural elements comprises two main steps:

1) generating an initial strut-and-tie model based on a stress field produced via elastic-plastic FEA for given boundary and loading conditions;
2) transforming such strut-and-tie model using non-linear optimisation methods to develop geometrically and statically admissible discrete stress fields.

The initial input consists of:

a) the geometric domain of a reinforced concrete structural element, such as a wall or a slab, including its width and the geometry of any internal openings;
b) the modulus of elasticity and yield strength of the materials;
c) the magnitude, direction, and point of application of the external forces;
d) the support conditions.

Based on this input, an elastic-plastic stress field is calculated through the Finite Element Method (FEM) to derive the tensor field of the stresses [16][17]. This model is then used to define an initial strut-and-tie model, which in turn becomes the input of the graphic statics workflow after being converted to an equivalent truss. The link between strut-and-tie modelling and graphic statics also makes it possible to encompass structural elements subject to bending moments and torsion whilst addressing both statically determinate and indeterminate structures [18]. The graphic static extension proposed by McRobie and Williams [19], where the edges of the form diagram can be subject not only to axial forces but also to bending moments and torsion, can also be used for the same scope.

It should be highlighted that the choice of the initial strut-and-tie model is left to the designer, and more than one interpretation of the stress field could be possible. That is, the designer is actively and creatively taking part in the process - based on the FEM input data - rather than being provided with the result of a black-box algorithm. Furthermore, only the topology of the initial truss is required and not its exact geometry in static equilibrium, as this can be generated directly by the ASF. In fact, the topology of the initial 2D truss on the x-y plane is lifted to 3D (x-y-z) to form a closed polyhedron. Since for a 2D truss to be in static equilibrium, it should be a projection of a flat-faced polyhedral ASF [11], this process automatically adjusts the coordinates of the vertices of the 2D truss by enforcing static equilibrium.

The polyhedral ASF of the 2D truss related to the strut-and-tie model (i.e. 2D form diagram) can then be reciprocated to a dual polyhedron. This polyhedron generates the reciprocal 2D force diagram when projected on the x-y plane. The 2D form and force diagrams can then be combined to create a Minkowski Sum, which can be further refined geometrically to an admissible discrete stress field. As discussed in [14], the derivation of an MS given a form and force diagram does not automatically translate into a geometrically compatible discrete SF. On the contrary, in the general case, it will not since there might be intersections between uniaxial stress fields. Furthermore, it is necessary to convert the nodal geometries to their compression-only equivalents [14].

At the same time, it is crucial to adjust the geometry of the resulting SF based on geometrical constraints such as the domain of the given concrete slab and the points of application of external forces. In the proposed approach, this is achieved by transforming the form ASF through a constraint-based optimisation. That is, the geometry of the ASF is parametrised and inputted to the gradient-based Low-storage Broyden–Fletcher–Goldfarb–Shanno (LBFGS) algorithm [20] implemented in the NLOPT library [21] to minimise the SF distance from the domain boundaries and openings. In this way, it can be ensured that the SF is entirely contained within the geometric boundary of the structural component. Furthermore, since the optimisation is performed on the flat-faced ASF, the strut-and-tie model is guaranteed to be in equilibrium without requiring iterative node-by-node reconstruction techniques of the force reciprocal. Moreover, for a given concrete strength, a scale factor for the force diagram and the wall width, and assuming constant stress design via the MS, it is possible to calculate the exact dimensions of the SF as well as the maximum applicable external load (i.e. collapse load).

3. Case studies

Figures 1 and 2 show two different solutions of the same case study – that of a wall with an opening – in terms of initial domain and loading (Fig. 1a, 2a). Firstly, given the stress data derived from elastic-plastic FEM analysis (Fig. 1b, 2b), two different topologies for the initial strut-and-tie geometry are proposed (Fig. 1c, 2c). Secondly, the struts and ties are identified (Fig. 1d, 2d). These are subsequently converted to equivalent self-equilibrated truss geometries by incorporating the external force as a truss member (Fig.1e, 2e). These constitute the initial form diagrams (Fig. 1f, 2f) which are lifted in 3D to form flat-faced ASFs. The ASFs are then reciprocated and projected back to 2D to produce the corresponding force diagrams (Fig. 1k, 2k). The pair of form and force diagrams are combined to produce the initial SF, which results in a geometrically and statically admissible rigid-plastic stress field (Fig. 1i, 2i) via the LBFGS algorithm and the adjustment of the form ASF. The stress field can be further refined geometrically so that the nodes are converted to their compression-only equivalents to accommodate the presence of rebars (Fig. 1j, 2j).

Figure 3 depicts the case of a beam with a variable cross-section subject to bending moment (Fig. 3a) for which the initial FEM (Fig. 3b) defines the starting point of the simplified strut-and-tie model (Fig. 3c). Similar to the example above, the strut-and-tie model is then converted to a self-equilibrated truss (Fig. 3e), the form ASF of which is parametrised and inputted to the LBFGS algorithm to produce a geometrically admissible SF (Fig. 3k). It should be highlighted that in this case, an additional geometrical constraint is provided by the top stress field generated between the opposing external forces of equal magnitude which should not intersect with the parallel field below (Fig. 3i). Lastly, the nodal geometries of the resulting rigid-plastic SF are further refined and converted to compression-only (Fig. 3j).

4. Discussion and conclusion

In this paper, the use of graphic statics-derived tools for the design of reinforced concrete elements is investigated. A novel method is presented combining pure graphic statics with preliminary structural analysis on the response of concrete structures using elastic-plastic stress fields. In particular, the initial strut-and-tie model sketched by the designer based on an elastic-plastic stress field is subsequently automatically adapted and verified by means of graphic statics, resulting in an admissible strut-and-tie model in static equilibrium. As a result, this approach allows designers to define realistic strut-and-tie models that satisfy both yield conditions and compatibility of deformations while providing interpretation, simplification, and refinement during this process. In this way, designers gain greater decision-making freedom and the ability to customize the design of reinforced concrete members.

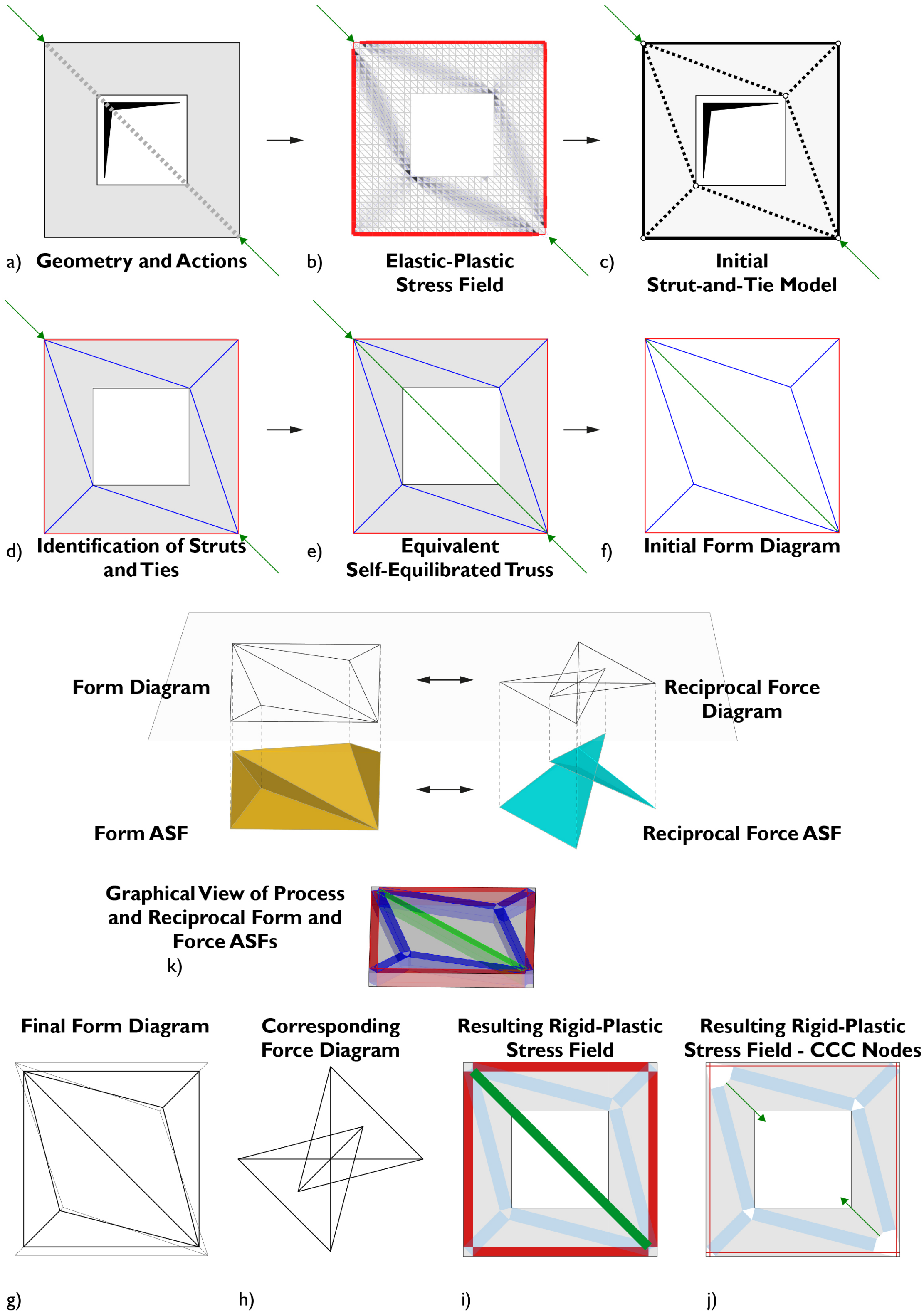

Figure 1: Case study 1: (a) geometry and actions; (b) Elastic-Plastic Stress Field; (c) initial strut-and-tie model; (d) identification of struts and ties; (e) self-equilibrated truss; (f) initial form diagram; (g) final form diagram; (h) corresponding force diagram; (i) resulting rigid-plastic stress field; (j) resulting rigid-plastic stress fields with converted compression-only nodes; and (k) graphical view of process and reciprocal form and force ASFs.

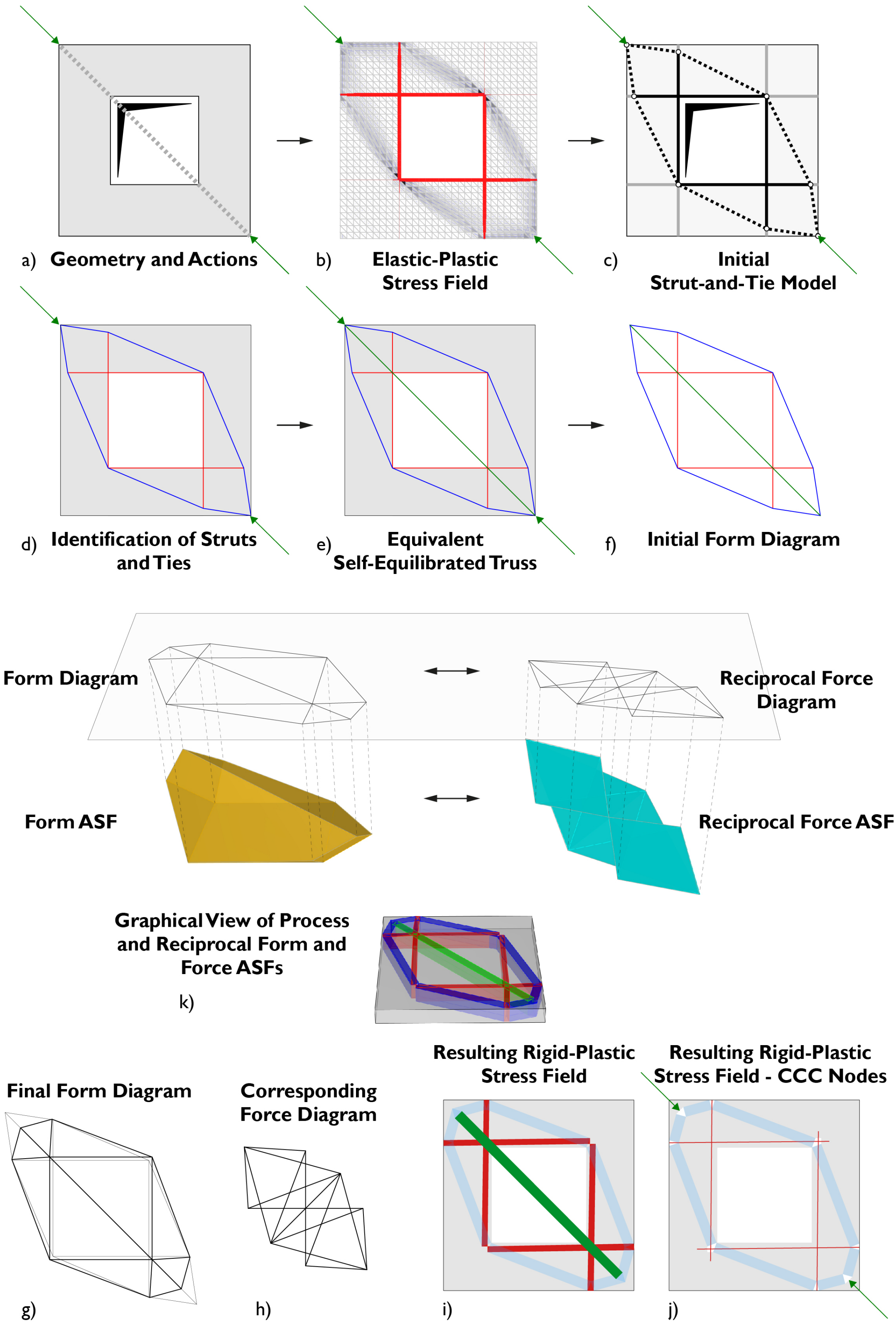

Figure 2: Case study 1: (a) geometry and actions; (b) Elastic-Plastic Stress Field; (c) initial strut-and-tie model; (d) identification of struts and ties; (e) self-equilibrated truss; (f) initial form diagram; (g) final form diagram; (h) corresponding force diagram; (i) resulting rigid-plastic stress field; (j) resulting rigid-plastic stress fields with converted compression-only nodes; and (k) graphical view of process and reciprocal form and force ASFs.

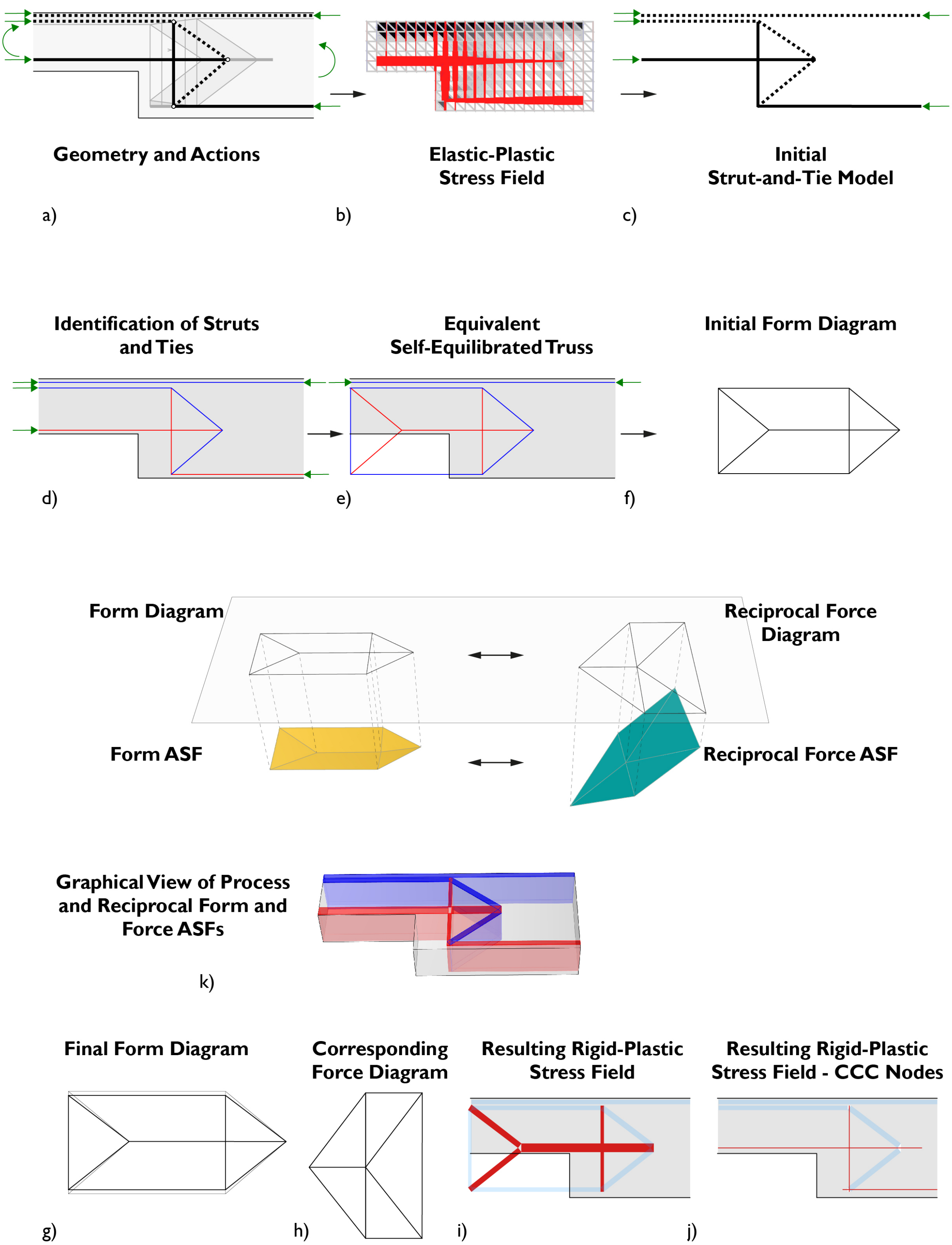

Figure 3: Case study 1: (a) geometry and actions; (b) Elastic-Plastic Stress Field; (c) initial strut-and-tie model; (d) identification of struts and ties; (e) self-equilibrated truss; (f) initial form diagram; (g) final form diagram; (h) corresponding force diagram; (i) resulting rigid-plastic stress field; (j) resulting rigid-plastic stress fields with converted compression-only nodes; and (k) graphical view of process and reciprocal form and force ASFs.

References

[1] M. Rankine, "Principle of the equilibrium of polyhedral frames," *Philosophical Magazine*, vol. 27, issue 92. 1864.

[2] J. C. Maxwell, "On reciprocal figures, frames and diagrams of forces," *Philosophical Magazine*, vol. 27, pp. 250-261, 1864.

[3] K. Culmann, *Die Graphische Statik,* Zurich: Meyer und Zeller, 1866.

[4] L. Cremona, *Le Figure Reciproche nella Statica Grafica,* Milano: Giuseppe Bernardoni, 1872.

[5] P. Block, J. Ochsendorg, "Thrust network analysis: a new methodology for three-dimensional equilibrium," *Journal of the International Association for Shell and Spatial Structures,* vol. 48, pp.167-173, Dec. 2007.

[6] M. Akbarzadeh, T. Van Mele and P. Block. "On the Equilibrium of Funicular Polyhedral Frames and Convex Polyhedral Force Diagrams," *Computer-Aided Design,* vol. 63, pp. 118-28, Jun. 2015.

[7] A. McRobie, "The Geometry of Structural Equilibrium," *Royal Society Open Science*, vol. 4, 2017.

[8] P. D'Acunto, J.-P. Jasienski, P. O. Ohlbrock, C. Fivet, J. Schwartz and D. Zastavni, "Vector- based 3D graphic statics: A framework for the design of spatial structures based on the relation between form and forces," *International Journal of Solids and Structures*, vol. 167, pp. 58-70, Aug. 2019

[9] P. O. Ohlbrock, P. D'Acunto, "A Computer-Aided Approach to Equilibrium Design Based on Graphic Statics and Combinatorial Variations", *Computer-Aided Design*, vol. 121, Apr. 2020

[10] J. C. Maxwell, "On Reciprocal Figures, Frames, and Diagrams of Forces", *Transactions of the Royal Society of Edinburgh*, vol. 7, pp. 160-208, 1870

[11] T. Mitchell, W. Baker, A. McRobie and A. Mazurek, "Mechanisms and states of self-stress of planar trusses using graphic statics, part I: The fundamental theorem of linear algebra and the Airy stress function," *International Journal of Space Structures*, vol. 31, no. 2-4, pp. 85-101, Aug. 2016.

[12] M. Konstantatou, P. D'Acunto and A. McRobie, "Polarities in structural analysis and design," *International Journal of Solids and Structures,* vol. 152-153, pp. 272-293, Nov. 2018.

[13] A. McRobie, "Maxwell and Rankine reciprocal diagrams via Minkowski sums for two-dimensional and three-dimensional trusses under load," *International Journal of Space Structures*, vol. 31, issue 2-4, pp. 203-216, Aug. 2016.

[14] M. Konstantatou, D'Acunto, A. McRobie and J. Schwartz, "Unified geometrical framework for the plastic design of reinforced concrete structures", *Structural Concrete*, vol. 21, issue 6, pp. 2320-2338, Sep. 2020.

[15] J. F. Rondeaux, P. D'Acunto, J. Schwartz and D. Zastavni, "Limit State Analysis of 2D Statically Indeterminate Networks Using Graphic Statics," *Proceedings of IASS Annual Symposia, IASS 2017 Hamburg Symposium: Graphic Statics*, pp. 1-10, 2017

[16] M. Fernández Ruiz, A. Muttoni, "On development of suitable stress fields for structural concrete", *ACI Structural Journal*, vol. 104, no. 4, pp. 495-502, Jul.-Aug. 2007.

[17] A. Muttoni, M. Fernández Ruiz and F. Niketic, "Design versus assessment of concrete structures using stress fields and strut-and-tie models," *ACI Structural Journal*, vol. 112, no. 5, pp. 605-615, Sep.-Oct. 2015.

[18] A. Muttoni, J. Schwartz and B. Thürlimann, *Design of Concrete Structures with Stress Fields*. Basel: Birkhäuser. 1997

[19] C. Williams and A. McRobie, "Graphic statics using discontinuous Airy stress functions", *International Journal of Space Structures*, vol. 31, issue 2–4, pp. 121–134, Jun. 2016.

[20] J. Nocedal, "Updating quasi-Newton matrices with limited storage," *Mathematics of Computation*, vol. 35, pp. 773-782, Jul. 1980.

[21] S. Johnson, "The NLopt optimisation package," *http://ab-initio.mit.edu/nlopt* (accessed 04/2017)